Solving Exponential Equations

An "exponential equation" is one where the variable you care about appears in the exponent.

The standard example is this: Given that $y = a \cdot b^x$, solve for x.

Exponential equations arise from processes involving repeated multiplication, such as growth and decay. Examples are commonly drawn from biology, economics, chemistry, electronics, and nuclear decay.

There are several ways of solving exponential equations, depending on what you know, what you want, and how much accuracy you need.

- If you need a symbolic solution, then logarithms are the right tool.
- If you have a specific problem and are allowed to use Excel, then Goal Seek is quick and effective when used properly. But be careful: to get an accurate answer, you need to solve for $\frac{Left\ Side}{Right\ Side} = 1$, not $Left\ Side - Right\ Side = 0$ as we've been doing for other problems.
- If you need a quick estimate of doubling time, then use Dr. Bartlett's "Rule of 70" or the economist's "Rule of 72".

Let's look more closely at each of these methods.

Symbolic Solution Using Logarithms

From our Cheat Sheet #2, recall these properties of logarithms:

1. $\log(a \cdot c) = \log(a) + \log(c)$ [log of a product]
2. $\log\left(\frac{a}{c}\right) = \log(a) - \log(c)$ [log of a fraction]
3. $\log(a^R) = R \cdot \log(a)$ [log of a power]
4. $\log\left(\left(a^R\right)^S\right) = S \cdot R \cdot \log(a)$ [log of power-to-a-power]
5. $\log_c(a) = \log_b(a) / \log_b(c)$ [change of base]
6. if $\log(a) = \log(b)$, then $a = b$

For most exponential equations, the best approach is to take logarithms of both sides, then use the properties of logarithms to tease out the variable you care about. Rule 3, log of a power, is particularly helpful.

For illustration, let's look at an example:

$4^{x-3} = 6^x$ given this equation, solve for *x*

$\log(4^{x-3}) = \log(6^x)$	take logarithms of both sides
$(x-3)\cdot\log(4) = x\cdot\log(6)$	expand using Rule 3, log of a power
$x\cdot\log(4) - 3\cdot\log(4) = x\cdot\log(6)$	distribute the multiplication
$x\cdot\log(4) - x\cdot\log(6) = 3\cdot\log(4)$	rearrange to get all *x* terms on left
$(\log(4)-\log(6))\cdot x = 3\cdot\log(4)$	factor to find the coefficient on *x*
$x = \frac{3\cdot\log(4)}{(\log(4)-\log(6))}$	divide both sides by the coefficient

So, the exact symbolic answer is $x = \frac{3\ln 4}{\ln 4 - \ln 6}$.

As a number, that value is about -10.25706775 .

Numeric Solution Using Goal Seek

In this class, we have traditionally solved an equation *f(x)* = *g(x)* by solving for *f(x)* – *g(x)* = 0. This works well when *f(x)* and *g(x)* have values that may include zero and where changing *x* adds or subtracts a significant amount to *f* or *g*. However, the values in exponential expressions often get very small. When they do, changing the exponent then adds or subtracts from *f* or *g* by only a correspondingly small amount. This presents a difficulty in solving for the exponent.

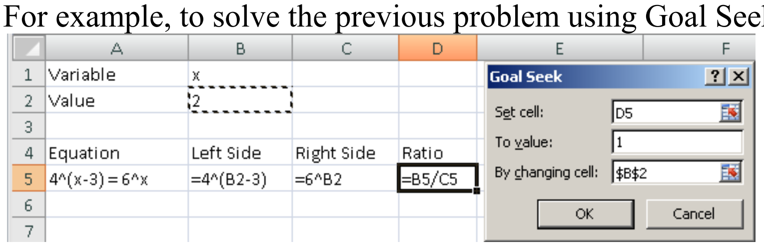
Consider for example solving $10^x = 10^{-20}$. The exact answer is of course *x* = -20. But any *x* less than -10 will give an equation whose left and right sides differ by less than 10^{-10}, a very small number! If you Goal-Seek for $10^x - 10^{-20} = 0$, then you'll probably get an answer that's way off.

Here is the solution to this difficulty:

> To accurately solve for possibly small but non-zero values of *f(x)* = *g(x)*, compute the ratio *f(x)* / *g(x)* , and Goal-Seek this ratio to value 1 .

This will solve the problem accurately regardless of how large or small *f(x)* and *g(x)* happen to be. Goal-Seeking for $10^x / 10^{-20} = 1$ from *x* = 0 gives *x* = -19.9999 even at default accuracy.

For example, to solve the previous problem using Goal Seek, you need to set up like this:



Given this setup, Goal Seek quickly converges to the correct answer of x ≈ -10.25702 .

	A	B	C	D
1	Variable	x		
2	Value	-10.25702		
3				
4	Equation	Left Side	Right Side	Ratio
5	4^(x-3) = 6^x	1.0435E-08	1.0435E-08	0.99998

On the other hand, trying to solve for Difference = 0 fails miserably, stopping with $x \approx -4$.

A Bit More About Logarithms...

There are two commonly used bases for logarithms: 10 and $e = 2.718282...$ (infinitely non-repeating).

Logarithms to the base *e* are often called "natural logarithms". For this reason, the function $\log_e$ (logarithm to the base *e*) is often named *ln*, while $\log_{10}$ (logarithm to the base 10) is often named *log*.

This convention is followed in Excel, where =LOG(100) returns the value 2 (since $10^2 = 100$), while =LN(100) returns the value 4.60517... (since $e^{4.60517...} = 100$). However, the convention varies from place to place, and it is always wise to "be sure you know what those symbols mean" before using them.

The number *e* arises naturally in growth problems. For example, if you invest $1 at 100% per year interest, compounded continuously, then at the end of a year you will have *e* dollars instead of 1 dollar. Natural logarithms have the additional useful property that $\log_e(1+v) \approx v$, when *v* is small.

Using Logarithms To Understand Bartlett's "Rule of 70"

Dr. Bartlett tells us that "constant (multiplicative) growth at N% per time period has a doubling time of $\frac{70}{N}$ periods". Why is that?

Well, if we have 1 unit of stuff at time 0, then we will have 1+N/100 units at time 1, $(1+N/100)^2$ units at time 2, $(1+N/100)^3$ units at time 3, and so on.

So, Dr. Bartlett's statement is really the same as asking this question:

For what value of *x* is $(1+N/100)^x = 2$?

Now we can understand how to answer that question.

If $(1+N/100)^x = 2$, then $\log_e((1+N/100)^x) = \log_e(2)$.

But $\log_e((1+N/100)^x) = x \cdot \log_e(1+N/100)$. Because $\log_e(1+v) \approx v$, $x \cdot \log_e(1+N/100)$ is approximately $x \cdot (N/100)$, when N/100 is small.

Hence, $x \cdot (N/100) = \log_e(2) \approx .693$, or $\boxed{x \approx .693 / (N/100) = 69.3/N}$.

The value 69.3 is hard to remember, and since the whole thing is only approximately correct anyway for N of a few percent, we round to 70 for convenience. Economists prefer rounding to 72 ("Rule of 72") because 72 has lots of small integer divisors. Both of them are pretty accurate. It turns out that 70 is actually the better number for N around 2-3%, while 72 is better for N around 7-8%, the historical rate of return for many investments.

Consumer Loans: Calculating the Number of Periods Using Logarithms

For completeness (take your aspirin!), here is a derivation of the consumer loan formula for number of periods, *N*.

$P = \frac{Lr(1+r)^N}{(1+r)^N - 1}$	Standard formula for payment.
$((1+r)^N - 1)P = Lr(1+r)^N$	Multiply both sides to kill the denominator.
$P(1+r)^N - P = Lr(1+r)^N$	Expand the left side.
$P(1+r)^N - Lr(1+r)^N = P$	Rearrange terms to put all $(1+r)^N$ on the left
$(P - Lr) \cdot (1+r)^N = P$	Factor the left side.
$(1+r)^N = \frac{P}{P - Lr}$	Divide by $(P - Lr)$ and simplify both sides.
$N \cdot log(1+r) = log\left(\frac{P}{P-Lr}\right)$	Take logarithms of both sides and apply Rule 3
$N = \frac{log\left(\frac{P}{P-Lr}\right)}{log(1+r)}$	Divide both sides by $log(1+r)$.

Amusingly, www.quickmath.com gives the answer as

$$N = -\frac{log\left(\frac{P-Lr}{P}\right)}{log(r+1)}$$

It's equivalent, of course.