Introduction to Equations
Math 101, Littlefield[1]

An <u>equation</u> consists of two expressions separated by an <u>equals sign</u> (=).

Examples include the following. (All are using single-letter variable names.)

1. $7 = 4 + 3$
2. $7x = 4x + 3x$
3. $\frac{7}{x} = \frac{4+3}{x}$
4. $6x + 15 = 27$
5. $I = prt$
6. $r = \frac{4.3 - AiR}{i}$
7. $6x + 1 = 6x$
8. $6 + 1 = 6$

Every equation is a statement that two expressions have the same value.

However, depending on the exact nature of the expressions, an equation can actually have several different meanings. (You figured this was coming, right?)

Let's look at the above examples in more detail.

1.	$7 = 4 + 3$	true, nothing more to be said
2.	$7x = 4x + 3x$	true for every possible value of x
3.	$\frac{7}{x} = \frac{4+3}{x}$	true for every value of x except x=0, when the expressions become undefined
4.	$6x + 15 = 27$	true for exactly one value of x, and false for all other values
5.	$I = prt$	for any values of p, r, and t, some value of I can be found to make it true.
6.	$r = \frac{4.3 - AiR}{i}$	similar to #5, except that when $i = 0$, the right side becomes undefined so that no value of r can be found to match up with values for A, i, and R.
7.	$6x + 1 = 6x$	false, for every possible value of x.
8.	$6 + 1 = 6$	false, nothing more to be said.

Sometimes mathematicians break these various meanings into groups and stick names on them:

- **Identity** — an equation that is true for all possible values of all variables.
- **Conditional** — an equation that is true for some values of variables.
- **Contradiction** — an equation that is never true, not for any assignment of values to variables.

The solutions or roots of an equation are the sets of values (if any) that make the equation be true.

In some cases, solutions are expressed as simple numbers.
For example, the solution to $6x + 15 = 27$ is $x = 2$.

In other cases, solutions are expressed as relationships between variables.
For example, the solutions to $ir = 4.3 - AiR$ can be written as $r = \frac{4.3 - AiR}{i}$. (If $i = 0$, then the first equation is a contradiction, and the second equation has an undefined right-hand side.)

Equivalent equations are equations that have exactly the same solutions.

The process of solving problems in algebra generally consists of starting with an equation that is easy to find or create, and turning it into an equivalent equation that is easy to use.

Generally speaking, if you do the same operation to both sides of an equation (both entire sides!) then you end up with an equivalent equation.

Some operations can add or remove solutions, and you need to be aware of this possibility. However, it is always safe to add, to subtract, and to multiply or divide by non-zero. Those operations are sufficient to solve all the problems that we'll be dealing with in the early parts of this course.

Solving simple equations

The following procedure will solve a huge number of equations — probably most equations that you will ever have to solve symbolically.

1. **Kill the denominators!** If either side of the equation is a fraction or sum of fractions, then pick one of the denominators and multiply both sides of the equation by it. If there are any denominators left, repeat the process. Or save steps by multiplying through by some common denominator all at once.
2. **Expand to eliminate the parentheses.** Fully expand both sides of the equation so you just have (at worst) a sum of products. Combine like terms if you want; it'll save some writing later.
3. **Isolate on one side all terms containing the variable you care about.** Pick the variable you want to solve for. Add and subtract as needed to move all terms involving that variable to one side of the equation, and all other terms to the other side. It's customary to put variable terms on the left.
4. **Find the coefficient on the variable.** Factor the variable side into (coefficient)*variable.
5. **Divide both sides by the coefficient.** This will leave you with just the variable on one side, and some typically ugly expression on the other side.
6. **Simplify the result.** Often the result expression will be a fraction that is not reduced to lowest terms, or there will be some terms lying around that can be combined, or something else that makes the expression look uglier than it needs to be. Fix those up if needed. This doesn't change the set of solutions, but it can make your answer more palatable to another human.

Example #1

Let's pick an equation that looks complicated, and solve it.

Here is a problem from electronics.

Given this equation:

$$r = \frac{4.3 - AiR}{i}$$

Solve for R .

Solution:

Step number	Equation	Explanation
1.	$r = \frac{4.3 - AiR}{i}$	Given.
2.	$ir = 4.3 - AiR$	Multiply both sides by the denominator.
3.	$AiR = 4.3 - ir$	Move all terms involving R to the left side, everything else to the right side.
4.	$(Ai) \cdot R = 4.3 - ir$	Factor the left side to find the coefficient (Ai).
5.	$\frac{(Ai) \cdot R}{(Ai)} = \frac{4.3 - ir}{(Ai)}$	Divide both sides by the coefficient.
6.	$R = \frac{4.3 - ir}{Ai}$	Simplify both sides and remove unnecessary parentheses. Done!

The one remaining step is to check our answer. (*"Mistakes happen..."*)

One really good way to check symbolic answers is by plugging in numbers. You can check for yourself (using Excel) that both the starting equation (1) and the ending equation (6) are satisfied by the following sets of numbers:

r	A	i	R
-35.3857	4	7	9
-26.14	3	5	9
-32.14	3	5	11

Since these numbers test every variable with at least two different "safe" values (not zero, not one, not any number that appears in the equation, and no two numbers the same), we can be pretty confident that the symbolic answer really is correct.

Example #2

Here's a different problem, this one from photography (optics).

Given this equation:

$$\frac{1}{f} = \frac{1}{o} + \frac{1}{i}$$

Solve for o .

Solution:

Step #	Equation	Explanation
1.	$\frac{1}{f} = \frac{1}{o} + \frac{1}{i}$	Given.
2.	$foi * \left(\frac{1}{f}\right) = foi * \left(\frac{1}{o} + \frac{1}{i}\right)$	Multiply both sides by *f*o*i*, to clear the denominators.
3.	$foi * \frac{1}{f} = foi * \frac{1}{o} + foi * \frac{1}{i}$	Expand to eliminate the parentheses.
4.	$oi = fi + fo$	Simplifying equation 3.
5.	$oi - fo = fi$	Rearrange, to get all the terms involving *o* on the left side.
6.	$io - fo = fi$	Reorder variables to put the one we care about at the end of each term. This makes it easier to identify the coefficient, in the next step.
7.	$(i - f) * o = fi$	Factor left side to identify the coefficient on *o*.
8.	$\frac{(i - f) * o}{(i - f)} = \frac{fi}{(i - f)}$	Divide both sides by the coefficient *(i-f)*
9.	$o = \frac{fi}{i - f}$	Simplify left side and eliminate unnecessary parentheses. This gives a good final answer.

The one remaining step is to check our answer. (*"Mistakes happen..."*)

Again, one really good way to check symbolic answers is by plugging in numbers. You can check for yourself (using Excel) that both the starting equation (1) and the ending equation (9) are satisfied by the following sets of numbers:

f	o	i
5	6.666667	20
6	7.5	30
3	4.285714	10

Because these numbers test every variable with at least two different "safe" values (not zero, not one, not any number that appears in the equation, and no two numbers the same), we can be pretty confident that the symbolic answer actually is correct.