Introduction to Algebra: The First Week

Background: According to the thermostat on the wall, the temperature in the classroom right now is 72 degrees Fahrenheit. I want to write to my friend in Europe, telling him this. But in Europe, they use a different scale for measuring temperature, called Celsius.

Question: How many degrees Celsius is 72 degrees Fahrenheit? How can we find out?

First, does anyone here happen to know the answer to this question? Nobody does? That's too bad — often the fastest way to get an answer to a specific question is to ask the people around you. (Of course, if someone claims to know the answer, you still have to decide whether to believe that they're correct. But that's a problem for another day.)

OK, how else can we find out? Ah, somebody remembers that there's a formula. Very good! Do we remember exactly what the formula is? No? Too bad.

How can we find the formula? Of course, "Ask the Web!"

So, what should we ask? We might try something like "*Fahrenheit to Celsius conversion*". Firing up Google, we quickly discover that "*fahrenheit to celsius*" is a common question — it's top on the list of standard completions as soon as we type "*Fah*". Apparently there's quite a bit of information on the Web about this topic, since Google reports around 3 million(!) results for that query. So, let's use that.

In response to the query "*fahrenheit to celsius*", Google returns a bunch of hits.

Most likely, the #1 hit will be to the "Fahrenheit to Celsius Converter" at http://www.wbuf.noaa.gov/tempfc.htm . When we visit that page, we discover that it is a bare-bones calculator — two cells named F and C, and nothing else. In particular, there's no formula. But still, we can at least use the calculator to answer the specific question that we started with. Typing the value 72 into the F field, we are rewarded with the value 22.22222222222222 in the C field. So now we know an answer to our question — 72 degrees Fahrenheit is 22.22-something degrees Celsius. (What's with all those 2's, anyway?) But we still have no clue how they did the calculation, and conceivably it could even be wrong.

In previous semesters, the #2 hit has gone to "Conversion of Temperature - Celsius to Fahrenheit" at http://www.mathsisfun.com/temperature-conversion.html . That page offers us a choice of tools and also a method: "**°F to °C: Deduct 32, then multiply by 5, then divide by 9"**. Let's try that. Deduct 32 from 72 gives 40. Multiply 40 by 5 gives 200. Divide 200 by 9 gives $\frac{200}{9}$, which can also be written as $22\frac{2}{9}$, or as $22.2\overline{2}$ (the bar means "repeat forever"), or approximately as 22.22222222222222 . So now we know where that 22.22-something comes from.

At this point, we also have a method to compute degrees Celsius for any other degrees Fahrenheit that we might want, and we don't even need continued access to the Web to do that. This is great.

But we still don't have a nice concise formula that anybody can understand. What we have is a method written in English.

We can, of course, rewrite the method to produce a formula ourselves. In standard algebraic notation, that gives this: $\boldsymbol{C = (F - 32) * 5/9}$.

It's interesting and important to notice that the preceding formula doesn't look like the formula that you will find in most books to convert Fahrenheit to Celsius: $\boldsymbol{C = \frac{5}{9}(F - 32)}$. The first formula is written in what we'll call "**Single Line Notation**". This is the dialect of algebra that is used for talking to calculators and computers — everything is on one line, and all the characters appear on standard keyboards. The second formula is written in "**Visual Notation**". This is the dialect of algebra that usually appears in textbooks and other publications. Visual Notation is very concise, but you can't use it with a computer. Converting back and forth between these dialects is a critical skill. We'll be doing it a lot.

Despite the differences in format, it's probably easy to convince yourself that both of the above formulas describe essentially the same calculation: subtract 32, then multiply by 5 and divide by 9.

However, if we dig deeper into how the mathsisfun.com calculator actually works, we discover something very interesting — it does not perform the same calculation! Instead, what the calculator does is this: $\boldsymbol{C = 100/(212 - 32) * (F - 32)}$. Not only does this calculation involve more operations than the others do, but most of the numbers are different — 5's and 9's versus 212, 100, and an extra 32.

Why should we have any faith that these apparently very different calculations are in some sense interchangeable??

The answer to this question speaks directly to those "**fundamental concepts and operations**" that are mentioned in the university catalog.

In essence, all three calculations are interchangeable because they always end up computing the same result, no matter what value you provide for $\boldsymbol{F}$. They are examples of "**equivalent expressions**". More formally, equivalent expressions are expressions that have the same value for all possible values of all variables that appear in them.

Now, we ordinarily think of an equation like $\boldsymbol{C = \frac{5}{9}(F - 32)}$ as telling us how to calculate a value for $\boldsymbol{C}$, given a value for $\boldsymbol{F}$. Sometimes we write $\boldsymbol{C(F) = \frac{5}{9}(F - 32)}$, to emphasize that $\boldsymbol{C}$ is a *function* of $\boldsymbol{F}$. (This notation is used a lot in advanced math like calculus.)

But we can also think of that equation as specifying what pairs of $\boldsymbol{F}$ and $\boldsymbol{C}$ "go together" in the sense that they make the equation true. For example, $\boldsymbol{F = 72}$ and $\boldsymbol{C = \frac{200}{9}}$ go together because it really is true that $\boldsymbol{\frac{200}{9} = \frac{5}{9}(72 - 32)}$. On the other hand, $\boldsymbol{F = 20}$ and $\boldsymbol{C = 45}$ do not go together because it is certainly not true that $\boldsymbol{45 = \frac{5}{9}(20 - 32)}$. Numbers or sets of numbers that go together to make an equation true are called "**roots**" of that equation. They are also said to "**satisfy**" the equation.

Thinking of equations this way opens up a lot more possibilities. We can, for example, think of equations like $\boldsymbol{\frac{9}{5}C = F - 32}$. Given specific values for both $\boldsymbol{F}$ and $\boldsymbol{C}$, it is easy to test whether the equation is true. But it may not be immediately obvious how to compute one of them, given the other.

Equations can be equivalent in the same sense as expressions. That is, "**equivalent equations**" are equations that are true for exactly the same values of all variables that appear in them. Speaking formally, they have the same roots.

The equations $\boldsymbol{C = (F - 32) * 5/9}$, $\boldsymbol{C = \frac{5}{9}(F - 32)}$, $\boldsymbol{\frac{9}{5}C = F - 32}$, and $\boldsymbol{F = \frac{9}{5}C + 32}$ are all equivalent in this sense. Whenever a pair of values ***C*** and ***F*** makes one of these equations true, that same pair makes all the other equations true also.

Most of classic symbolic algebra is concerned with methods to transform one expression or equation into a different expression or equation that is **equivalent** but is somehow **more convenient**. For example, if we know that $C = \boldsymbol{\frac{5}{9}(F - 32)}$, then algebra provides a set of methods that allow us to figure out that $\boldsymbol{F = \frac{9}{5}C + 32}$.

These methods are important because they allow many equations to be created in one form that is convenient to write, and then converted to another form that is equivalent but is more convenient to use.

For example, if we know that $C = \boldsymbol{\frac{5}{9}(F - 32)}$, but what we really need is to find ***F*** given ***C***, then algebra lets us figure out that $\boldsymbol{F = \frac{9}{5}C + 32}$, and now we have a simple recipe for doing the computation — multiply ***C*** by 9, divide by 5, and add 32. This latter equation is said to be "solved for ***F***" because ***F*** appears by itself on one side of the equation and not at all on the other side. Another way of saying the same thing is that we now have a "formula for ***F***".

In the case of Fahrenheit-Celsius conversion, there is arguably little point in using algebra because we can probably find the formula for ***F*** at the same place that we found the formula for ***C***. Someone else already did the algebra for us.

The real power of algebra comes in solving problems where we can't just look up the formula we really want, or where we have to write all the equations ourselves as part of solving some other problem.

OK, classical algebra is all about rearranging symbols, how to turn one equation into another equation that is equivalent but is somehow more useful. Frequently this is done to find a number that we care about — to "find a numeric solution".

But some equations cannot be solved by algebra. And even when equations can be solved, it is very easy to make mistakes in manipulating the symbols, so that in the end the symbolic mistakes produce the wrong numbers. How can we avoid or work around these problems?

It turns out that computation – working directly with numbers – can help with both problems.

We can use computation to check our symbol manipulation by seeing if numbers that make one equation true also make a different equation true. If they do not, then the two equations definitely are <u>not</u> equivalent, and if they were supposed to be equivalent, then we know we made a mistake.

We can also use computation to find numbers that satisfy an equation, even if the equation is too complicated to solve by algebra.

In class, we walked through the process of constructing an Excel spreadsheet that illustrates both of these aspects. Showing all the equations, it looked like this:

	P	Q	R	S
1	Variables	C	F	
2	Values	22.2222222222222	72	
3				
4	Equations	Left Side	Right Side	Difference
5				
6	C = (F-32) * 5 / 9	=Q2	=(R2-32)*5/9	=Q6-R6
7	9/5 * C = F - 32	=9/5*Q2	=R2 - 32	
8	F = 9/5 * C + 32	=R2	=9/5 * Q2 + 32	

and showing just values, it looked like this:

	P	Q	R	S
1	Variables	C	F	
2	Values	22.22222222	72	
3				
4	Equations	Left Side	Right Side	Difference
5				
6	C = (F-32) * 5 / 9	22.22222222	22.22222222	0
7	9/5 * C = F - 32	40	40	
8	F = 9/5 * C + 32	72	72	

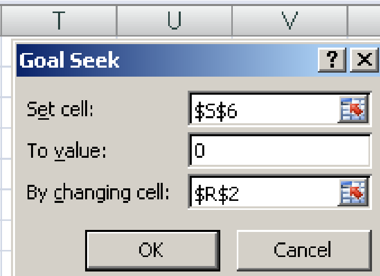
Notice that the equations in this spreadsheet are written in yet another dialect of mathematics: single line notation with explicit multiplication, with variables named using row-column notation such as R2 (column R, row 2).

Rows 6-8 of this spreadsheet show three different equations that are supposed to be equivalent, and we can see that indeed, they are all satisfied by the same pair of values, F=72 and C=22.222222222 . We also saw (not shown here), that all three equations are <u>not</u> satisfied by F=86 and C=22.22222222, but again <u>are</u> satisfied by F=86 and C=30.

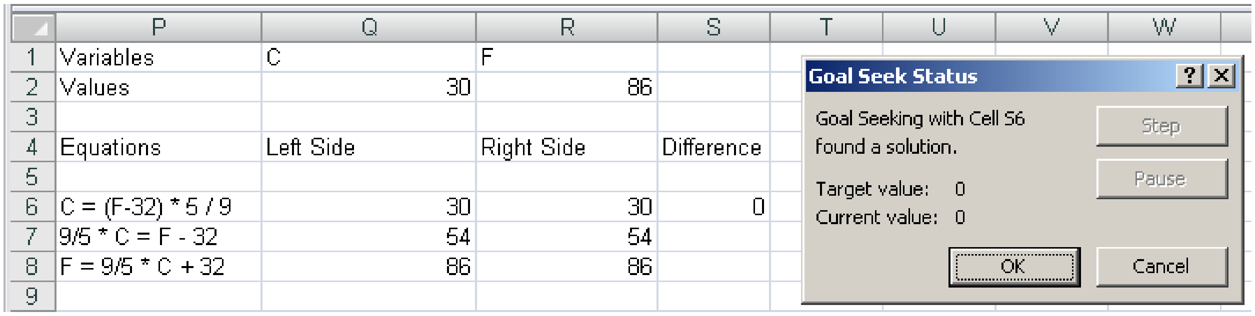
And finally, we saw that we can ask Excel to solve the equation in row 6, ***C*** = (***F***-32)*5/9, for the specific case of C=30. This was done by just by plugging in C=30 and asking Excel to adjust F as needed to make the two sides of the equation have the same value, that is, the difference between the sides = 0.

Here is what the request looks like. Notice that at this instant, none of the equations is satisfied, since C=30 and F=72 do not go together.

	P	Q	R	S
1	Variables	C	F	
2	Values	30	72	
3				
4	Equations	Left Side	Right Side	Difference
5				
6	C = (F-32) * 5 / 9	30	22.22222222	7.777778
7	9/5 * C = F - 32	54	40	
8	F = 9/5 * C + 32	72	86	
9				



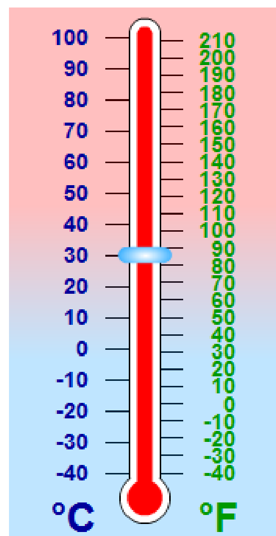
When we click OK, Excel figures out for us what value of F is required to make ***C*** = (***F***-32)*5/9, when ***C*** = 30. The screen updates to say this:



	P	Q	R	S
1	Variables	C	F	
2	Values	30	86	
3				
4	Equations	Left Side	Right Side	Difference
5				
6	C = (F-32) * 5 / 9	30	30	0
7	9/5 * C = F - 32	54	54	
8	F = 9/5 * C + 32	86	86	
9				

Just that simple, Excel has figured out for us that ***F*** = 86 is what's required. No algebra needed. And this same method works even for a lot of other problems that algebra simply cannot solve at all. This is powerful stuff!

Closing out the Fahrenheit to Celsius problem, let's take a moment to ask "Why are those formulas what they are?" Perhaps the best way to understand this is to go back to that little "Interactive Thermometer" presented on the mathsisfun page. It looks like this.



Notice that on the left we have a scale for degrees Celsius, and on the right we have a scale for degrees Fahrenheit. You can't tell exactly from this graphic, but the graphic does illustrate several key facts about the two temperature scales:

- When the tick marks on one side are evenly spaced, then the tick marks on the other side are evenly spaced also.
- 0 degrees Celsius is exactly equal to 32 degrees Fahrenheit (by definition), and
- 100 degrees Celsius is exactly equal to 212 degrees Fahrenheit (also by definition).

It is the combination of these three things that produces a unique one-to-one relationship between degrees Celsius and degrees Fahrenheit. There are lots of ways to represent that relationship by formulas. Four equally valid representations are:

$$\boldsymbol{C = \frac{5}{9}F - \frac{160}{9}}$$

$$\boldsymbol{C = \frac{5}{9}(F - 32)}$$

$$\boldsymbol{C = 100/(212 - 32) * (F - 32)}$$

$$\boldsymbol{C = ((F - 32)/(212 - 32)) * (100 - 0) + 0}$$

The first of these is in "standard form for a linear equation", *y=mx+b,* with m=5/9 and b= -160/9. But although this would be the standard form in one context, it is wildly inappropriate for temperature conversion. I don't recall ever seeing this form outside of an algebra class. The second form is what usually appears in textbooks. The third form, you may recall, is what appears in the mathsisfun calculator. It computes the correct numbers, but quite frankly, it's a bit weird. On the other hand, the fourth form, which you have not seen before, is arguably the most natural form for many linear relationships because it directly contains the matching values for two corresponding points: 0 degrees C goes with 32 degrees F, and 100 degrees C goes with 212 degrees F. A similar equation for any other linear relationship can be written by just substituting appropriate values in place of 0, 32, 100, and 212. I use this form frequently when writing software to do coordinate conversions for computer graphics applications.

In all, we have spent the better part of two full class periods looking at this apparently silly little problem of Fahrenheit to Celsius temperature conversion. Is it worth all that time? Yes! Think back to that entry in the Course Catalog, describing Math 101: **"Fundamental algebraic operations and concepts"**. Well, it turns out that we have seen most of them in this one problem.

Here is the list that was shown on the summary slide:

- A "**variable**" is just a name that stands in for a number we don't know yet.
- An "**equation**" is a statement that two numbers are equal, even though we don't know what they are!
- Equations also define sets of numbers that "go together" to satisfy the equations.
- A "**formula**" is a special kind of equation: variable = (other stuff not involving that same variable)
- "**Equivalent expressions**" look different but compute the same numbers.
- "**Equivalent equations**" look different but are satisfied by the same sets of numbers.
- **Algebra** manipulates <u>symbols</u>, to turn equations that are convenient to write or to look up into equivalent equations that solve the problem you care about.
- One good way to check your algebra: plug in numbers, and make sure that the numbers work.
- "**Model problem**": you already know the answer, use that to check your method.
- "Black box" calculator: numbers in, numbers out, no clue what goes on in the middle.
- **Spreadsheets** can solve problems by finding <u>numbers</u> that satisfy equations, even without doing any algebra!
- Dialects of notation: English words versus algebraic symbols, "visual" versus "single line".

And at the bottom of that slide was another one of the big points of the course:

There is no one "Language of Mathematics", there are a bunch of dialects.
Be sure you know what those symbols <u>mean</u>!